MATHEMATICAL LOGIC HOMEWORK 2

Due Monday, February 12.

Problem 1. *Let $\mathcal{L}$ be the language $\{+; 0\}$ and consider the structure $\mathcal{R}$ with universe $\mathbb{R}$ where $+$ is interpreted as the usual addition and 0 as zero. Show that there is no formula $\phi(v, w)$, such that for all $a, b \in \mathbb{R}$, $\mathcal{R} \models \phi(a, b)$ if and only if $a < b$. [Hint: Find an $\mathcal{L}$-isomorphism not preserving $<$.]*

We say that K is an elementary class of models if there is a theory T, such that $K = \{\mathcal{M} \mid \mathcal{M} \models T\}$.

Problem 2. *1) Let $\mathcal{L} = \{E\}$, where E is a binary relation. Let T_0 be the axioms for equivalence relations. Suppose $T \supset T_0$ is an $\mathcal{L}$-theory, such that for all n, there is $\mathcal{M} \models T$ with an equivalence class of size at least n. Then there is $\mathcal{M} \models T$ with an infinte equivalence class.*

2) Prove that the class of all equivalence relations where every class is finite is not an elementary class.

Problem 3. *Let $\mathcal{L} = \{R\}$ where R is a binary relation. Recall that a graph is an $\mathcal{L}$-structure $\mathcal{G}$, where $R^{\mathcal{G}}$ is symmetric and irreflexive. We say that a graph is connected if for each distinct x, y, we can find a finite path from x to y. Prove that the class of connected graphs is not an elementary class.*

Problem 4. *Let $\mathcal{L}$ be the language with one binary relation symbol $<$. Let T be an $\mathcal{L}$- theory extending the theory of linear orders such that T has infinite models. Show that there is $\mathcal{M} \models T$ and an order preserving embedding $j : \mathbb{Q} \to M$ of the rational numbers into M. For example there is $\mathcal{M} \models Th(\mathbb{Z}; <)$, in which the rational order embedds.*

[Hint: add constant c^q for $q \in \mathbb{Q}$ and sentences $c^q < c^r$ for all $q < r$ to T.]

Problem 5. *We say that an ordered abelian group $(G; +; 0; <)$ is archimedean if for all $x, y \in G$ with $x > 0, y > 0$, there is a positive integer m such that $x < my$. Show that there are non-archimedean models of $Th(\mathbb{Q}; +; <)$.*